3.1 Determinants (Introduction)

Key idea: The notion of determinant we saw for a 2x2 matrix can be extended to any nxn matrix. Much like in 2x2 setting, det(A) tells us a lot about an nxn matrix A, namely if A is invertible or not. We will see in section 3.3 the geometric information det(A) gives about the transformation $\vec{x} \longmapsto A\vec{x}$.

Recall that a 2x2 matrix is invertible if it is row-equivalent to I_2, so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{a \neq 0}{\sim} \begin{bmatrix} a & d \\ 0 & d - bc/a \end{bmatrix} \sim \begin{bmatrix} a & 0 \\ 0 & ad-bc \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so long as $ad-bc \neq 0$.

i.e., so long as $\det(A) \neq 0$.

We now extend the definition of det(A) to all nxn matrices and study this quantity in detail. (Note: the book (pg 166) gives an algebraic justification for higher dimensional determinants much like we did for 2x2. The algebra is a touch nasty, so we omit this.)

Def: Let $A = [a_{ij}]$ be an nxn matrix, we define the determinant of A, written det(A) as follows: if $n=1$, $\det(A) = \det([a]) = a$.

if $n \geq 2$, define A_{ij} to be the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^n a_{1n}\det(A_{1n})$$

Notice this is an alternating sum of determinants for smaller matrices. To compute det(A), we need first know $\det(A_{1j})$ and so on. In practice, this involves a lot of computation but we will see methods to make calculation much easier.

Ex) Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$A_{11} = \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \qquad A_{13} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$\det(A) = 1 \cdot \det(A_{11}) - 5\det(A_{12}) + 0\det(A_{13})$$

So we need find A_{11}, A_{12}, A_{13} and their determinants

For brevity we write $|A|$ in place of $\det(A)$:

$$|A_{11}| = \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} = 0-2 = -2, \quad |A_{12}| = \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 0-0=0, \quad |A_{13}| = \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = -4.$$

So $|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| = 1(-2) - 5(0) + 0(-4) \boxed{= -2}$

Notice how the presence of 0's in A simplify the calculation, the next fact allows us to fully exploit the structure of A to aid in finding determinants.

Def: For an $n \times n$ matrix A, the (i,j)-cofactor of A is: $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Fact: The determinant of A is equal to the sum of the cofactors across any row or column, i.e.

for any i: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{n1}C_{n1}$

and,

for any j: $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$.

Note: We call such a sum a cofactor expansion across the i^{th} row (j^{th} column). The sum is always alternating with minus signs determined by a_{ij} position in A i.e.

$$\begin{array}{c} a_{11} \\ a_{12} \\ a_{13} \\ \vdots \end{array} \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \end{bmatrix}$$

This follows from the def. of C_{ij}:

$$(-1)^{i+j} = \begin{cases} +1 & i+j \text{ even} \\ -1 & i+j \text{ odd.} \end{cases}$$

Ex Recompute $\det A$ for

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{array}{l} \leftarrow a_{31} \; + \\ \leftarrow a_{32} \; - \\ \leftarrow a_{33} \; + \end{array}$$

using a cofactor expansion down the third column.

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (0)\overset{+1}{(-1)^{3+1}}\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} + (-1)\overset{-1}{(-1)^{3+2}}\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + (0)\overset{-1}{(-1)^{3+3}}\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= 0 + (-2) + 0 \boxed{= -2} \checkmark$$

We conclude with another example of an informed choice of a cofactor expansion. We will see that triangular matrices are particularly nice to compute the determinant of.

Ex Compute $\det(B)$ where
$$B = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$
We choose a cofactor expansion involving many zeros: column 1.

$$\det(B) = 3C_{11} + 0C_{21} + 0C_{31} + 0C_{41} + 0C_{51}$$
$$= 3(-1)^{1+1}\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3\cdot\left(2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}\right)$$

another cofactor expansion

$$= 3\cdot 2\cdot\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$
$$= 3\cdot 2\cdot(-2)$$ ← from above
$$= \boxed{-12}.$$

Notice that as we expand down the first and second columns, we simply multiply a smaller determinant by the diagonal entries.
If every column of B was like the first two, the determinant would have simply been the product of the diagonal entries.

Fact: If A is a triangular matrix, then $\det A$ is the product of the diagonal entries of A.

On a numerical note: the number of calculations it takes to compute a determinant of a random $n\times n$ matrix is roughly $n!$ ... which is huge. To compute a 25×25 determinant requires 1.5×10^{25} operations. So, if we do a trillion operations a second, we'll need 500,000 years.

So how does Maple compute this in less than a second? There are some properties of $\det(A)$ that severely ease our computational task, and we heavily exploit that last fact.